Noetherian Rings—Dimension and Chain Conditions

Abhishek Banerjee
57/1/C, Panchanantala Lane
Behala, Calcutta 700034, INDIA

Received: March 23, 2005      Accepted: September 16, 2005

ABSTRACT

In this paper we look at the properties of modules and prime ideals in finite dimensional noetherian rings. This paper is divided into four sections. The first section deals with noetherian one-dimensional rings. Section Two deals with what we define a “zero minimum rings” and explores necessary and sufficient conditions for the property to hold. In Section Three, we come to the minimal prime ideals of a noetherian ring. In particular, we express noetherian rings with certain properties as finite direct products of noetherian rings with a unique minimal prime ideal, as an analogue to the expression of an artinian ring as a finite direct product of artinian local rings. Besides, we also consider the set of ideals $I$ in $R$ such that $M \neq IM$ for a given module $M$ and show that a maximal element among these is prime. In Section Four, we deal with dimensions of prime ideals, Krull’s Small Dimension Theorem and generalize it (and its converse) to the case of a finite set of prime ideals. Towards the end of the paper, we also consider the sets of linear dependencies that might hold between the generators of an ideal and consider the ideals generated by the coefficients in such linear relations.

I. ALL RINGS ARE ASSUMED TO COMMUTATIVE RINGS WITH IDENTITY

Our main purpose is to study the noetherian rings of finite dimension and their modules. In this respect we shall prove results on the decomposition of a noetherian ring as a direct product of simpler noetherian rings and also consider if and only if conditions on the finitely generated modules of the ring that determine the dimension of the ring. We will also take up separately the question of dimension of a prime ideal in relation to the number of its generators and also establish a converse to these results [1-4].

Although artinian rings are defined apparently in a dual fashion (descending chain condition) to noetherian rings (ascending chain condition), it turns out that the artinian rings are just the noetherian zero dimensional rings (see [5]). Therefore, we shall skip the case of dimension zero and start Section 1 with noetherian one-dimensional rings.

II. SECTION 1

We start with 1-dimensional noetherian rings. Suppose that $M$ is a finitely generated $R$-module. We will consider the support of $M$, i.e., the maximal ideals $m$ such that $M_m = 0$. Our first proposition will deal with the case of noetherian one-dimensional rings. Then we have the following result.

Proposition: Let $R$ be a domain. The following statements are equivalent:

1) $R$ is noetherian and 1-dimensional.
2) If $M$ is a finitely generated $R$-module, $M$ is either of finite length of $\text{Supp}(M) = \text{Spec}(R)$. (The Support $\text{Supp}(M)$ of a module is the set of all
prime ideals \( p \) of \( R \) such that \( M_p \neq 0 \). The Spec of a ring refers to a set of all prime ideals of the ring.)

Proof: Let \( R \) be noetherian and 1-dimensional. Let \( M \) be a finitely generated \( R \)-module. We take a filtration of \( M \) such that the quotients of consecutive submodules in it are isomorphic to some \( R/p \). If all of them are of the form \( R/m \) for some maximal ideal \( m \), \( M \) comes out to be of finite length. If any of them is not maximal, it must be zero (since the ring is 1-dimensional). As such, one of the quotients is isomorphic to \( R \). This quotient cannot become 0 under any localization whatsoever and hence \( \text{Supp}(M) = \text{Spec}(R) \).

Conversely, let us assume (2). Take a non-zero ideal \( I \) in \( R \). Then \( R/I \) is finitely generated. However, if we localize with respect to the prime ideal \( 0(R \ \text{a domain}) \), we have generated \( (R/I)_0 = 0 \). Thus, \( \text{Supp}(R/I) \) is not the same as \( \text{Spec}(R) \). Thus \( R/I \) is of finite length and hence noetherian. Now suppose that there is an infinite increasing chain of ideals in \( R \): say \( I_0 \subset I_1 \subset I_2 \subset \ldots \). Then if we take a non-zero ideal from this chain, say \( I_i \), we see that \( R/I_i \) is non-noetherian. This is a contradiction. Hence, \( R \) is noetherian. Now, we take a non-zero prime ideal \( p \) in \( R \). Consider \( R/p \) which is of finite length as an \( R/p \) module. Thus \( R/p \) is an artinian ring. Hence 0, which is a prime ideal in \( R/p \), is maximal. Thus, \( p \) is a maximal ideal. Hence \( R \) is 1-dimensional.

We now consider another equivalent condition for 1-dimensional noetherian domains. Accordingly, we have the following result.

Proposition: Let \( R \) be a domain. Then the following conditions are equivalent:
1) \( R \) is noetherian and one-dimensional.
2) Every finitely generated \( R \)-module of non-zero annihilator is of finite length.

Proof: Let \( M \) be an \( R \)-module of non-zero annihilator \( I \). Then \( M \) may be considered as an \( R/I \) module. Since \( R/I \) is a 0-dimensional noetherian ring, \( M \) is of finite length.

Conversely, we consider the case \( M = R/J \), where \( J \) is a non-zero ideal, as an \( R \)-module. We see that it has non-zero annihilator, i.e. \( J \), and hence must be of finite length. Thus \( R/J \) is artinian, i.e. noetherian and 0-dimensional. Hence, \( R \) is noetherian. Also, since \( R \) is a domain, \( (0) \) is a prime ideal. Thus \( R \) is 1-dimensional.

III. SECTION 2: ZERO MINIMUM RINGS

Let \( R \) be a noetherian ring. Then there might exist infinite descending chains of ideals in \( R \). As an intermediate between noetherian and artinian rings, we will allow the descending chain condition to creep into the noetherian ring in a restricted manner. Consider all the ideals which may be written as the intersection of such an infinite descending chain. If \( R \) is non-artinian, this is a non-empty set. Let \( J \) be a maximal element of this set. Then \( R/J \) has a curious property. Any infinite descending chain of ideals in this ring has zero intersection. We now define:

Definition: Zero minimum Ring. A ring \( R \) is said to be a zero minimum ring if the intersection of an infinite decreasing chain of ideals in \( R \) is zero. (Note that \( R \) is not assumed to be noetherian.)

The following proposition is an analogue of the standard theorem which says that \( R \) is artinian. \( R \) is noetherian and 0 dimensional. However, here we have the condition that \( R \) is a domain. To see similar blends of chain conditions, the reader may consult [6].

Proposition: Let \( R \) be a domain. Then the following are equivalent:
1) \( R \) is noetherian and 1-dimensional.
2) \( R \) is a Zero-minimum Ring.

Proof: Suppose that \( R \) is noetherian and 1-dimensional. Take an infinite descending chain of ideals in \( R \) and consider their intersection, say \( J \). If \( J \) is not 0, \( R/J \) is a noetherian ring having all its prime ideals maximal (the minimal prime ideal 0 has been eliminated by taking \( R/J \)). Hence \( R/J \) is an artinian ring and we have a decreasing infinite chain in \( R/J \). This is clearly a contradiction.

Conversely, we now assume that \( R \) is a zero minimum ring. Then each \( R/J \) is artinian, for non-zero \( J \). Thus, each \( R/J \) is noetherian for non-zero \( J \), and by the same argument as in the previous result, this gives us that \( R \) is noetherian. Also, this tells us
that all non-zero prime ideals in \( R \) are maximal. Since \( R \) is a domain, we have one more prime ideal, namely 0, which makes the ring 1-dimensional.

The idea of studying noetherian 1-dimensional rings by studying their artinian quotient rings may also be traced back to [7]. In this respect, the reader may also consider the more recent [8]. The previous proof also says that if \( R \) is not a domain and at the same time is a zero minimum ring, we would have had: \( R \) is a noetherian ring and all non-zero prime ideals are maximal. Since 0 is not a prime ideal in \( R \), we have:

Proposition: If \( R \) is a zero minimum ring implies that it is artinian (and thus there is no infinite decreasing chain at all).

Proof: We see that \( R \) is a noetherian ring with all prime ideals maximal. Hence, \( R \) is artinian.

We will use this to prove the next result:

Proposition: Let \( R \) be a noetherian ring (and non-artinian) and let \( J \) be maximal among the ideals that can be written as the intersection of infinitely long, decreasing chains of ideals in \( R \). Then \( J \) is prime.

Proof: Consider \( R/J \). Note that \( R/J \) is noetherian but not artinian (there exists an infinite chain intersecting to 0). But \( R/J \) is a zero intersection ring. From the previous result, if \( R/J \) is not a domain, it must be artinian. Hence \( R/J \) is a domain. Thus, \( J \) is a prime ideal.

We end this section with the following theorem.

Proposition: If \( R \) is a zero minimum ring with non-zero Jacobson radical (the Jacobson radical is the intersection of all the maximal ideals of \( R \)), \( R \) must have finitely many maximal ideals.

Proof: Suppose that \( R \) has infinitely many maximal ideals. Choose countably many of these maximal ideals, say \( m_1, m_2, \ldots \). Consider the following chain of ideals:

\[ m_1 \supset m_1 \cap m_2 \supset m_1 \cap m_2 \cap m_3 \supset \ldots \]

The inclusions are strict because: if there is equality at the \( k^{th} \) stage, \( m_k = m_1 \cap m_2 \cap \ldots \cap m_{k-1} \). This will imply that \( m_k \) contains some other maximal ideal, which is absurd. Thus, there is an infinite chain. Since \( R \) is a zero-minimum ring, the intersection of this chain is \((0)\). But, the intersection contains the Jacobson radical \( J \). Thus, \( J = 0 \).

IV. SECTION 3: MINIMAL PRIME IDEALS

We shall now try to see the properties of the minimal prime ideals of a noetherian ring. It is well known that a noetherian ring has finitely many prime ideals, (see [5]). We shall use this property heavily in subsequent discussion. Further we note the following: if \( R \) is a noetherian ring and \( I \) and ideal in \( R \), the ring \( R/I \) is also noetherian. Thus, it will also have finitely many prime ideals, i.e. there are only finitely many prime ideals minimal over a given prime ideal \( I \). We shall take the liberty of referring to the prime ideals minimal over a given ideal \( I \) as the minimal primes of \( I \) rather than the minimal primes of \( R/I \).

A remarkable fact about the artin ring is that it can be written as a finite direct product of artin local rings (see [5]). This is proved as follows (and the proof proceeds along the lines of the Chinese Remainder Theorem): The artin ring \( R \) has only finitely many maximal ideals, say \( m_1, \ldots, m_n \) and the Jacobson radical \( J \) is nilpotent, say \( J^k = 0 \). Consider the product \( \Pi m_i^k \). Since the \( m_i \) are mutually co-prime, this product equals \( \cap m_i^k \). But \( \Pi m_i^k \subseteq (\cap m_i)^k = (0) \). Hence we have the natural isomorphism from \( R = R/\cap m_i^k \) to the direct product of the artinian local rings \( R/m_i^k \). We note that in a 0-dimensional ring we might actually think of the maximal ideal as a minimal ideal. We would like to see some version of this property for higher dimensional rings as well. However, for this we see in the following proposition that we must make the assumption that the minimal prime ideals of the ring are actually co-prime. (The property of being mutually co-prime is obvious for the maximal ideals, but it has to be assumed when we are dealing with minimal prime ideals.)

Proposition: Let \( R \) be a finite dimensional noetherian ring such that the minimal prime ideals in \( R \) are actually co-prime. Then, if \( R \) is \( k \)-dimensional, \( R \) can be written as a finite direct product of noetherian \( k \)-dimensional rings, each having only one minimal prime ideal.
Proposition: Let $R$ be a noetherian ring and $M$ be a finitely generated non-zero $R$-module. Consider the set of all ideals $J$ in $R$ such that $M 
eq JM$. This set is obviously non-empty since $M 
eq 0$. Consider an ideal $J$ maximal with respect to this property. Then $J$ is prime.

Proof: Consider the module $N = M/JM$. This may be treated as an $R/J$ module. Now suppose that $J$ is not prime. Consider any prime ideal $P$ containing $J$; then $P$ contains $J$ strictly. But this means that $M = PM$. However, $M = pM$ implies that $M_p = 0$, applying Nakayama's lemma to $M_p$ over $R_p$. Thus $(M/JM)_p = 0$ for each prime ideal $p$. But then $M/JM = 0$ or $M = JM$. This is a contradiction.

V. SECTION 4: Krull's Theorem (Small Dimension Theorem) and Generalizations

We now have come to the legendary Small Dimension Theorem of Krull. It gives us an upper bound for the height of a prime ideal in a noetherian ring based on the number of its generators. We state the theorem as follows:

Small dimension Theorem. Let $I$ be an ideal in a noetherian ring $R$ and let $p$ be a prime ideal in $R$ minimal over $I$. Then if $\mu(I)$ denotes the number of generators for $I$, $\mu(I) \geq ht(p)$.

We refer the reader to [6] for a proof of the above theorem. We note that the above theorem tells us that the height of any prime ideal $p$ in a noetherian ring is finite (we apply the theorem with $I = p$). Further, this means that if the ring has only finitely many maximal ideals, then the supremum of the heights of the maximal ideals is also finite. This supremum is referred to as the dimension (or Krull dimension) of the ring. The theorem therefore shows that the krull dimension of a semi-local noetherian ring is finite.

We consider next the extension of the small dimension theorem (and its converse) to finite sets of prime ideals. Let $p_1, \ldots, p_n$ be a set of prime ideals with no order relations among them. This means that none of the prime ideals $p_i$ is contained in any of the primes $p_j, i \neq j$. Let their heights be $r_1, \ldots, r_n$, respectively; and let $r = \max(r_1, \ldots, r_n)$. If $I$ is an ideal generated by $k$ elements such that each $p_1, \ldots, p_n$ is minimal over $I$, the $k \geq r$. This is obvious. Now we consider the converse. The converse of the small dimension theorem is as follows: Given a prime ideal of height $r$, there exists an ideal $I$, generated by $r$ elements, such that $p$ is minimal over $I$. Again we refer the reader to [6] for a proof of the theorem. Accordingly, we replace $p$ by a finite set of prime ideals with no order relations among them. Now we claim that there should exist an ideal $I$ generated by $r$ elements such that each $p_i$ is minimal over it. Here $r$ is the maximum of the heights of the prime ideals $p_i$.

Proposition: Let $p_1, \ldots, p_n$ be a set of prime ideals in a noetherian ring with no order relations among them, i.e. none of the prime ideals $p_i$ is contained in $p_j$ for any $i \neq j$. Let $ht(p_i) = r_i$ and let $r = \max(r_1, \ldots, r_n)$. Then there exists an ideal $I$ generated by $r$ elements such that each prime of the prime ideals $p_i$ is minimal over $I$, i.e. the image of $p_i$ is a minimal prime ideal of $R/I$.

Proof: We shall apply induction on $r$. If $r = 0$, each of the prime ideals $p_i$ is a minimal prime ideal and hence we can take $I = (0)$, which is generated by 0 elements. Now, we assume the result to be true for $r - 1$.

Let $q_1, \ldots, q_s$ be the minimal prime ideals of $R$ ($R$ being noetherian, this set is finite). Some of the $p_i$'s may be minimal prime ideals of $R$, let us assume that $q_1, \ldots, q_s$ are the minimal prime ideals among them. It is possible that $s = 0$. Take any $p_i$. Then $p_i$ cannot be a subset of $\cup_{j=s+1}^{n} q_j$ (by prime avoidance). Take $a_i \in$
$p_i = \cup_{i=n+1}^d q_i$. Let $a = a_1a_2...a_n$. Consider the
ring $R(a)$. Each of the prime ideals $p_i$
contains $a$. But, neither of the prime ideals
$q_{n+1},..., q_d$ contains $a$ since they do not
contain any of $a_1,...,a_n$.

Consider those $p_i$ for which $ht(p_i) = r$. Each $p_i$
contains a minimal prime ideal and these minimal prime ideals cannot occur among the $q_{n+1},...,q_d$ because the $p_i$’s have no
order relations among them. The minimal prime ideals contained among the $p_i$’s of height $r$ all come from the ideals $q_{n+1},...,q_d$.
Since neither of the prime ideals $q_{n+1},...,q_d$
contain $a$, in the ring $R(a)$, these prime ideals are excluded. Hence, the height of
those primes $p_i$ such that $ht(p_i)$ (in $R$) gets
reduced by at least one. Hence, the height of
each $p_i$ in $R(a)$ is $\leq r - 1$. Assume that
$max(ht(p_1),...,ht(p_n)) = k < r$ (in the ring
$R(a)$). By induction, we will have an ideal $I$
in $R(a)$ generated by $k$ elements such that
each $p_i$ is minimal over it. Thus, in the ring
$R$, each $p_i$ will be minimal over the ideal
generated by $I$ and $a$. This ideal has $k+1$
generators. Since $ht(p_i) = r$ for at least one
$p_i$, $r \leq k + 1$ and we know that $k < r$. Thus $k = r+1$ and the ideal we have is generated by
exactly $r$ elements.

Take a prime ideal of height $r$ in a
noetherian ring. Then there exists an ideal $I$
generated by $r$ elements such that $p$ is
minimal over $I$. Let $I = (a_1,...,a_r)$. We want
to know more about the generators $a_i$. More
precisely, we want to know what relations
they satisfy. If $I$ is not free on $a_1,...,a_r$, then exist (non-trivial)
linear combinations of the $a_i$’s which are 0. Let $I_1$
be the ideal generated by the coefficients of $a_i$ in all
such combinations. More precisely, consider all linear combinations $\sum x_ia_i$ of the $a_i$’s that are 0. Consider all the $x_i$’s appearing in these, that is the coefficients of $a_i$. Let us designate the ideal generated by
these $x_i$’s as $I_1$. Similarly, consider the $x_i$’s appearing in these combinations, that is, the
coefficients of $a_2$. Let us designate the ideal generated by
these $x_i$’s as $I_2$. Similarly, we define $I_3,...,I_r$. We note that:

Proposition: The sum of ideals $I_1 + I_2 +...+ I_r$
is contained inside $p$.

Proof: We note that the height of $p$ in $R$ is
the same as its height in $R_p$. Also, $p^*$ will
remain a minimal prime ideal over $I^*$ in $R_p$.
Thus $I^*$ should have at least $r$ generators. If

$I_1$ is not contained in $p$, $I_1^*$ contains 1 and
hence, in $R_p$, we can write $a_i/1$ as a linear
combination of $a_j/1,...,a_r/1$. Hence $I_1 \subseteq p$
and so also for $I_2,...,I_r$. Thus $I_1 + I_2 +...+ I_r$
is contained in $p$.

Corollary: If $a_i$ is a generator for the ideal $I$,
Ann($a_i$) $\subseteq p$.

Having considered the prime ideals
minimal over a given ideal $I$, we come to the
problem of determining the number of prime
ideals minimal over a given ideal $I$. This is
the same as determining the number of prime
ideals minimal over a given radical
ideal for $p$ contains $I$ iff $p$ contains
the radical of $I$.

Proposition: Consider the products of prime
ideals contained inside $I$, i.e. products of the form $p_1p_2...p_n$ contained in $I$, where $p_i$ need
not be all distinct, although each $p_i$ contains $I$.
Consider such a product having minimal
number of distinct terms, say $k$ terms. Then
there are at most $k$ many prime ideals
minimal over $I$.

Proof: Suppose that $p_1p_2...p_n$ is the chosen
product and it has $k$ distinct terms. Consider
a prime ideal $p$ containing $I$. If each of the
$p_i$’s has an element not contained in $p$, say
$x_i$, we see that the product $x_1x_2...x_n$
though contained in $I$, is not contained in $p$.
Thus $p$ $\not\subset p_i$ for some $i$. If $p$ is chosen to be minimal
over $I$, we see that it must equal $p_i$. Thus $I$
has at most $k$ prime ideals minimal over it.

ACKNOWLEDGEMENTS

The author is grateful to the referees for
raising several interesting concerns,
suggesting appropriate references and for
suggesting improvements in the style of the
paper.

REFERENCES

1. David Eisenbud, Craig Huneke, and
Bernd Ulrich, “A simple proof of some
generalized principal ideal theorems,”

2. David Eisenbud, E. Graham Evans, Jr.,
“A generalized principal ideal theorem,”
53.


---

**INDIAN STATISTICAL INSTITUTE**

Calcutta, New Delhi, Bangalore and Hyderabad

The Indian Statistical Institute is a unique institution devoted to the research, teaching and application of statistics, natural sciences and social sciences. Founded by professor P. C. Mahalanobis in Kolkata on December 17, 1931, the institute gained the status of an Institution of National Importance by an act of the Indian Parliament in 1959.

What began a small room in the Presidency College in 1931 now comprises buildings on several acres of land in four major cities (Calcutta, New Delhi, Bangalore, and Hyderabad). What began with a total annual expenditure of less than 250 Rupees in 1931 now has a total annual expenditure of over 15,000,000 Rupees. What began in 1931 with a solitary human ‘computer’ working part-time, now comprises over 250 faculty members and over 1,000 supporting staff and several modern-day personal computers, workstations, mini-computers, super-mini-computers, and mainframe computers. Impressive as these figures are, they convey little idea of the road traversed, the range of activities undertaken and the intimate relationship of the institute with the life of India.

http://www.isical.ac.in/
Why Sigma Xi?
THE HONOR SOCIETY FOR SCIENTISTS AND ENGINEERS

Gain professional credentials
Being active in Sigma Xi demonstrates to employers and graduate schools that you have achieved a high standard of excellence. As a member of this prestigious society, you join the company of over 180 Nobel Prize winners and other distinguished scientists and engineers.

Receive research funding
Twice annually, the Sigma Xi Grants-in-Aid of Research program awards research grants to nearly 1,000 graduate and undergraduate students. As a Sigma Xi member you will be eligible to apply for grants from all available funds.

Build your professional network
Gain valuable leadership experience and make career contacts through local chapters. Sigma Xi provides a unique platform for interacting within a multi-disciplinary community of researchers, educators, administrators and others, both within and outside your institution.

Become a lifelong learner
Stay current in all fields with American Scientist, Sigma Xi's award-winning magazine. As a member benefit, your American Scientist subscription is a valuable resource that offers articles in a variety of disciplines, resources, and a comprehensive book review section.

Contribute to the scientific community
Sigma Xi members form a powerful international network of over 500 institutions. Through individual members and chapters, Sigma Xi addresses ethics, education, advocacy, and other issues affecting scientific society.

Have fun
Build community and a sense of belonging with other scientists at your institution. Get involved in a special event with the help of others at your local chapter.

Sigma Xi, The Scientific Research Society • 99 Alexander Drive • B.0. Box 13075 • Research Triangle Park, NC 27709 • 919-548-4691 • 800-243-6500

www.sigmaxi.org